A Comprehensive Study of Elliptic Integrals and Elliptic Functions

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Shubham Damahe
(MS23MTH001)

under the supervision of

Dr. Purnima Satapathy

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur - 440010 (India)
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Department of Mathematics
Visvesvaraya National Institute of Technology, Nagpur

Declaration

I, Shubham Damahe, hereby declare that this dissertation titled"A Comprehensive Study of Elliptic Integrals and Elliptic Functions" is carried out by me at the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other institution/University.

Shubham Damahe

Date:

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This is to certify that the dissertation entitled "A Comprehensive Study of Elliptic Integrals and Elliptic Functions" is a bonafide workdone by **Mr Shubham Damahe** (Enrollment No. MS23MTH001). The project work has been completed under my guidance and supervision in a satisfactory manner. This work is a part of the requirement for the award of **Master of Science in Mathematics** from Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. This dissertation fulfills the requirements relating the nature and standard of work for the award of Master of Science degree.

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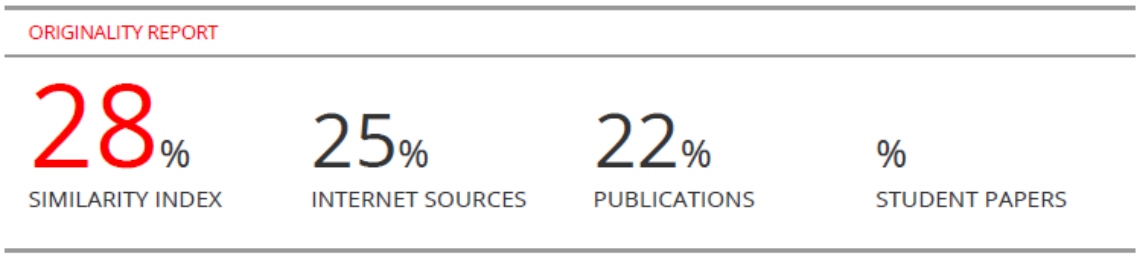

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Abstract

Elliptic Integrals arise naturally in a wide range of mathematical and physical problems where classical techniques fall short due to the presence of square roots of polynomials of degree three or four. This thesis explores the theory, classification, and applications of elliptic integrals, with a focus on their role in computing arc lengths of curves such as the lemniscate of Bernoulli, the motion of pendulums beyond the small-angle approximation, and the analysis of nonlinear systems. We begin with a historical overview and foundational definitions, followed by the reduction of general elliptic integrals into the standard forms: those of the first, second, and third kinds. Techniques for evaluating these integrals, including transformations, series expansions, and numerical methods, are discussed in detail. Special attention is given to the geometric interpretation of elliptic integrals through their connection to elliptic functions and modular forms. We will examine the comparison between Jacobi elliptic functions and Trigonometric functions, focusing on their properties such as periodicity and angle addition formulas. Finally, several classical and modern applications are presented, highlighting the enduring significance of elliptic integrals in both pure and applied mathematics.

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Chapter 1

Introduction

Introduction to Elliptic Integrals

Elliptic integrals arise naturally in problems involving arc lengths of ellipses, hence their name. These are a class of integrals that cannot, in general, be expressed in terms of elementary functions. A typical form of an elliptic integral is:

$$\int R(x, \sqrt{P(x)})\, dx,$$

where R is a rational function of its arguments and $P(x)$ is a polynomial of degree 3 or 4 with no repeated roots.

Historically, elliptic integrals were studied in the context of computing the arc length of an ellipse. For example, the arc length s from the origin to a point on an ellipse can be written as:

$$s = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta,$$

which is known as the incomplete elliptic integral of the second kind.

There are three standard types of elliptic integrals:

- **Elliptic integral of the first kind:**

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

which represents the amplitude φ as a function of arc length u.

- **Elliptic integral of the second kind:**

$$E(\varphi,k) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\theta}\, d\theta,$$

which appears in the computation of arc lengths and energy expressions in mechanics.

- **Elliptic integral of the third kind:**

$$\Pi(n;\varphi,k) = \int_0^{\varphi} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}},$$

which arises in more advanced applications like geodesics and motion in central force fields.

Elliptic integrals play a fundamental role in many areas of mathematics and physics, including the study of pendulum motion, wave propagation, electromagnetism, and the theory of elliptic functions. They are invertible, and their inverses are the Jacobi elliptic functions, which generalize trigonometric functions to elliptic geometry.

1.1 *A Brief History of Elliptic Integrals*

Origins and Early Developments

- **Ancient Roots**: The study of elliptic integrals began with the problem of calculating the arc length of an ellipse. Unlike circles, ellipses do not yield simple formulas for arc length, leading to integrals involving square roots of polynomials.
- **17th Century**: During the development of calculus, mathematicians such as Isaac Newton and James Stirling encountered integrals arising from arc lengths of conic sections and other curves, recognizing their complexity.

18th Century Progress

- **Giulio Fagnano (1714)**: Studied the arc length of the lemniscate (a figure-eight shaped curve) and discovered addition formulas and symmetries. This early work hinted at the deep structure of elliptic integrals.

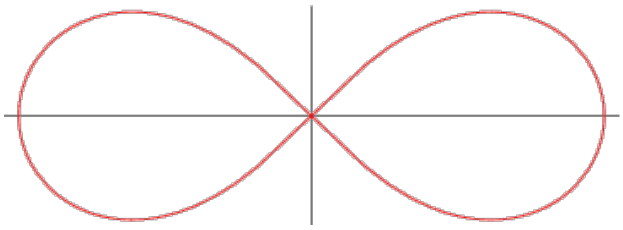

Figure 1.1: Lemniscate

- **Leonhard Euler**: Extended Fagnano's results and helped formalize the integrals arising in problems of arc length. Euler began recognizing general classes of such integrals.

Legendre's Canonical Forms (Late 18th – Early 19th Century)

- **Adrien-Marie Legendre**: In his monumental work *Exercices de Calcul Intégral* (1811–1816), he systematically studied elliptic integrals and classified them into three standard forms:
  - **First kind**:
$$F(\phi,k) = \int_0^{\phi} \frac{1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta$$
  - **Second kind**:
$$E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\,d\theta$$
  - **Third kind**:
$$\Pi(n;\phi,k) = \int_0^{\phi} \frac{1}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}\,d\theta$$
- He also created extensive numerical tables for their computation.

Shift to Elliptic Functions

- **Niels Henrik Abel and Carl Gustav Jacobi (1820s–1830s)**: Introduced a revolutionary viewpoint by inverting elliptic integrals to define *elliptic functions*, such as Jacobi's sn, cn, and dn.
- This shift led to significant developments in complex analysis, algebraic geometry, and modular forms.

Modern Significance

- Elliptic integrals are now fundamental in many fields:
  - **Physics**: Pendulum motion, electromagnetism, general relativity.
  - **Engineering**: Signal processing, wave propagation.
  - **Mathematics**: Geometry, number theory, complex analysis.
- **Arc length of Lemniscate**

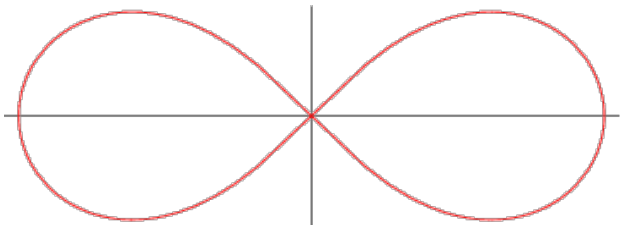

Figure 1.2: Lemniscate curve

The lemniscate was discovered by Jakob Bernoulli in 1694. He expressed its equation as:

$$(x^2+y^2)^2 = 2a^2(x^2-y^2)$$

In polar coordinates, this becomes:

$$r^2 = 2a^2\cos(2\theta) \tag{1.1}$$

Let us simplify by choosing $a = \frac{1}{\sqrt{2}}$, which transforms equation (1.1) into:

$$r^2 = \cos(2\theta)$$

To compute the total arc length L of the lemniscate, we use the formula for arc length in polar coordinates:

$$L = 4\int_0^{\frac{\pi}{4}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Given that $r^2 = \cos(2\theta)$, the expression simplifies to:

$$L = 4\int_0^{\frac{\pi}{4}} \sqrt{\cos(2\theta)}\, d\theta$$

Now, using the transformation $\cos(2\theta) = \cos^2(\psi)$, the above integral becomes:

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2(\psi)}\, d\phi$$

This can be rewritten as:

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{2\cos^2(\psi)+\sin^2(\psi)}\, d\psi$$

Next, let $t = \cos(\psi)$. Then, we obtain:

$$L = 4\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

Bernoulli's study of the lemniscate's arc length served as a cornerstone for the future advancement of elliptic function theory.

- **Elastic curve**

Elastic Curve (Bernoulli, 1691): A slender, flexible rod is deformed so that its two ends form right angles with a specified line H (Figure1.3).

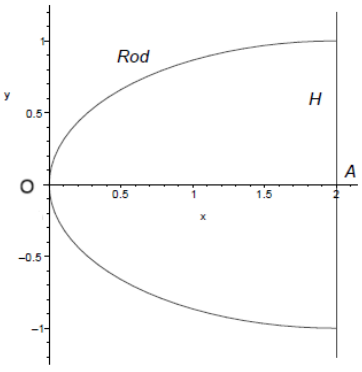

Figure 1.3: Elastic curve

Here a denotes length of OA

Bernoulli was able to show that the upper half of the curve is given by the equation

$$y = \int_0^x \frac{t^2}{\sqrt{a^4 - t^4}} dt \tag{1.1}$$

where $0 < x < a$. Assuming $a = 1$ simplifies the analysis, but it also means we lose track of the actual length of the rod. In fact, equation (1.1) shows that the arc length from the origin to a point (x, y) on the scaled elastic curve is:

$$l(x) = \int_0^x \frac{1}{\sqrt{1 - t^4}} dt$$

half length of the rod is

$$l(x) = \int_0^1 \frac{1}{\sqrt{1 - t^4}} dt$$

- **Equation of ellipse**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

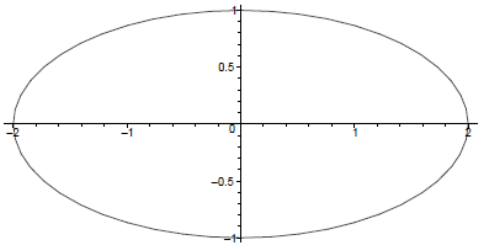

Figure 1.4: Ellipse $x^2 + \frac{y^2}{4} = 1$

The complete arc length is four times the length of the segment located in the first quadrant, where a specific relationship holds.

$$y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}$$

$$y'(x) = -\frac{xb}{a^2\sqrt{1 - (\frac{x}{a})^2}}$$

Thus we obtain

$$l(a,b) = 4\int_0^a \sqrt{1+y'(x)^2}dx$$

substituting the value of $z = \frac{x}{a}$ in above equation

$$= 4a\int_0^1 \sqrt{\frac{1-ez^2}{1-z^2}}dz$$
$$= 4a\int_0^1 \frac{1-ez^2}{\sqrt{(1-ez^2)(1-z^2)}}dz,$$

Which is elliptic integral of second kind. In the succeeding chapters Elliptic integrals of first kind, second kind and third kind are discussed extensively

Chapter 2

First kind

2.1. Introduction to the Elliptic Integral of the First Kind

Elliptic integrals arise naturally in the study of arc lengths of ellipses and appear widely in physics, engineering, and number theory. Among these, the *elliptic integral of the first kind* plays a central role, especially in problems involving periodic motion such as the simple pendulum. This section explores its formulation, transformation under substitution, its series representation and different physical examples with their solutions

The *incomplete elliptic integral of the first kind* is defined as:

$$u = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

Introducing the substitution $t = \sin\theta$, it follows that $y = \sin\varphi$, and the integral transforms into:

$$u = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\varphi, k)$$

where:

- ϕ is the *amplitude*,
- k is the *modulus*, with $0 \leq k < 1$.

This integral cannot, in general, be expressed in terms of elementary functions. It arises naturally when computing the arc length of an ellipse and is also the inverse function of the Jacobi elliptic function:

$$F(\phi, k) = u \quad \Leftrightarrow \quad \phi = \mathrm{am}(u, k).$$

2.1.1. Special Case: $k = 0$

When $k = 0$, the integral simplifies to the elementary form:

$$u = \int_0^y \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} y.$$

Thus, the inverse relationship is:

$$y = \sin u$$

This special case connects elliptic integrals to classical trigonometric functions, demonstrating that elliptic integrals generalize the inverse sine function.

2.1.2. Complete Elliptic Integral of the First Kind

When $y = 1$, or equivalently $\varphi = \frac{\pi}{2}$, the integral becomes:

$$K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

This is referred to as the *complete elliptic integral of the first kind*, denoted $K(k)$.

Hence, one can define:

$$\mathrm{am}(K) = \frac{\pi}{2}$$

2.1.3. Series Expansion

The integrand in $F(\varphi, k)$ may be expanded using a binomial series:

$$(1-k^2\sin^2\theta)^{-1/2}$$

Substituting this expansion into the integral yields a power series in k^2:

$$F(\varphi,k) = \varphi + \frac{1}{2}k^2V_2 + \frac{1\cdot 3}{2\cdot 4}k^4V_4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6V_6 + .., \tag{2.1}$$

where:

$$V_{2n} = \int_0^{\varphi} \sin^{2n}(\theta)\, d\theta$$

This expansion is especially useful in approximating elliptic integrals for small k.

2.1.4. Evaluation at $\varphi = \frac{\pi}{2}$ using Wallis' Formula

When $\varphi = \frac{\pi}{2}$, the integrals V_{2n} may be evaluated using *Wallis' formula*:

$$\int_0^{\frac{\pi}{2}} \sin^{2n}(\theta)\, d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{\pi}{2}$$

Substituting these into the earlier equation 2.1:

$$\frac{2}{\pi}K(k) = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \cdots$$

This is a series expansion of $K(k)$ in even powers of k, commonly used in numerical computations and analysis.

2.2. Jacobi Elliptic Function

Abel, who, around 1823, analyses that if one study the integral u as a function of x as follows,

$$u = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},$$

then the difficulty of studying is similar as that of trigonometric functions by taking u as a function of x in

$$u = \int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

In both cases, the inversion of the integral leads to functions with richer, more tractable properties-

namely, the circular functions in the trigonometric case, and the Jacobi elliptic functions in the elliptic case.
The three primary **Jacobi elliptic functions** are:

- $\mathrm{sn}(u,k)$: sine amplitude
- $\mathrm{cn}(u,k)$: cosine amplitude
- $\mathrm{dn}(u,k)$: delta amplitude

Here, u is the argument (kind of like the "angle " in trigonometry) and k is the modulus of the elliptic functions. Rather than analyzing the integral by treating x as the independent variable, Abel suggested examining the upper limit x as a function of u.
Later, Jacobi introduced the concept of the amplitude of u, denoted by $\phi = \mathrm{am}(u)$, where ϕ is called the amplitude. Treating x as a function of u, we have:

$$x = \sin\phi = \sin(\mathrm{am}(u))$$

Alongside, this function we have two more relations are:

$$\cos\phi = \cos(\text{am}(u)), \quad \Delta\phi = \Delta(\text{am}(u)) = \sqrt{1-k^2\sin^2\phi}$$

Gudermann (who was Weierstrass's teacher), in *Crelle's Journal*, Vol. 18, p. 12, suggested simplifying the notation by defining the following:

$$x = \sin\phi = \text{sn}(u),$$

$$\sqrt{1-x^2} = \cos\phi = \text{cn}(u),$$

$$\sqrt{1-k^2x^2} = \Delta\phi = \text{dn}(u)$$

Remark: In the Jacobian elliptic functions $\text{sn}(u,k)$, $\text{cn}(u,k)$, and $\text{dn}(u,k)$, the variable u serves a role analogous to an angle in trigonometric functions, while the parameter k (known as the modulus) influences the shape of the functions, much like the eccentricity defines the shape of an ellipse.

Identities

$$sn^2u + cn^2u = 1$$

$$dn^2u + k^2sn^2u = 1$$

2.3. Examples of First kind

- **The Arc length of lemniscate**: The equation of curve is

$$(y^2+x^2)^2 + a^2(y^2-x^2) = 0;$$

writing $x = rcos\theta, y = r\sin\theta$, the above equation becomes

$$r^2 = a^2cos2\theta$$

From the expression $ds^2 = dr^2 + r^2d\theta^2$, the differential of arc is

$$ds = \pm\frac{dr}{\sqrt{1-\frac{r^4}{a^4}}} = \pm\frac{ad\theta}{\sqrt{1-2\sin^2\theta}}$$

writing $r = a\cos\phi$, so that $2\sin^2\theta = \sin\phi$, it is seen that

$$S = a\int_0^\theta \frac{d\theta}{\sqrt{1-2\sin^2\theta}} = \frac{a}{\sqrt{2}}\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}} = \frac{a}{\sqrt{2}}F\left(\frac{1}{\sqrt{2}},\phi\right)$$

A quadrant of the lemniscate is

$$S = a\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{1-2\sin^2\theta}} = \frac{a}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}} = \frac{a}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right)$$

we can easily calculate this integral when 'a' is known.

- **Time period of pendulum:** $T = \sqrt{\frac{a}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \sqrt{\frac{a}{g}}F(k,\frac{\pi}{2})$

  where a is the length of pendulum and g is the gravity

The motion of a simple pendulum (a mass swinging from a fixed point under gravity) is described by the differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

where: - θ is the angular displacement, - g is the acceleration due to gravity, and L is the length of the pendulum.

For small angles ($\theta \approx \sin\theta$), this simplifies to simple harmonic motion with a period:

$$T_0 = 2\pi\sqrt{\frac{L}{g}}$$

However, for large amplitudes, the equation is nonlinear and requires the use of an elliptic integral. The exact period of the pendulum is:

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

where k is the elliptic modulus, given by:

$$k = \sin\frac{\theta_0}{2}$$

and θ_0 is the maximum angular displacement.

Connection to the Complete Elliptic Integral of the First Kind:

$$T = T_0K(k) \quad \text{where} \quad K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$K(k)$ is the complete elliptic integral of the first kind.

For small θ_0, $K(k) \approx \frac{\pi}{2}$, recovering the small-angle approximation $T \approx T_0$.

Example
Table 1 gives value of the integrals

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\sin^2\theta\sin^2\phi}}$$

For Example if we take $\theta = 79^\circ 30'$ then K=3.10640

Table 2 gives value of the integral

$$F(k,\phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1-\sin^2\theta\sin^2\phi}}$$

For Example if we take the value $\theta = 65^\circ$ and $\phi = 81^\circ$ then $F(k,\phi) = 1.94377$

With the help of Table-1,Table-2 one can easily find out the value of given elliptic integral of first kind when ϕ and θ are known

2.4. Addition Theorem

2.4.1. Euler's Addition theorem

The addition theorem for the first kind is a formula that relates the value of the elliptic integral at the sum of two arguments to the values at the individual arguments. This is particularly useful in the theory of elliptic functions.
Let us take $f(x) = (1-x^2)(1-k^2x^2)$
Then

$$\int_0^x \frac{dt}{\sqrt{f(t)}} + \int_0^y \frac{dt}{\sqrt{f(t)}} = \int_0^z \frac{dt}{\sqrt{f(t)}} \tag{2.2}$$

where

$$z = \frac{x\sqrt{f(y)} + y\sqrt{f(x)}}{1-k^2x^2y^2} \tag{2.3}$$

Proof : Let us consider the equation

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = 0 \tag{2.4}$$

Obviously, that (2.4) defines a level set of function $z = z(x,y)$ defined by (2.2)
If we take

$$u = \int_0^x \frac{dt}{\sqrt{f(t)}}$$

THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

θ	K	E	θ	K	E	θ	K	E
0°	1 57080	1.57080	50°	1.93558	1.30554	82° 0′	3.36987	1.02784
1	092	068	51	5386	29628	12	9457	670
2	127	032	52	7288	28695	24	3.41994	558
3	187	1.56972	53	9267	27757	36	4601	447
4	271	888	54	2.01327	26815	48	7282	338
5	379	781	55	3472	25868	83 0	3.50042	231
6	511	650	56	5706	24918	12	2884	126
7	668	495	57	8036	23966	24	5814	023
8	849	296	58	2.10466	23013	36	8837	1921
9	1 58054	114	59	3002	22059	48	3.61959	821
10	284	1.55889	60	2.15652	21106	84 0	3.65186	1.01724
11	539	640	61	8421	20154	12	8525	628
12	820	368	62	2.21319	19205	24	3.71984	534
13	1 59125	073	63	4355	18259	36	5572	443
14	457	1.54755	64	7538	17318	48	9298	354
15	814	415	65	2.30879	16383	85 0	3.83174	266
16	1.60198	052	66	4390	15455	12	7211	181
17	608	1.53667	67	8087	14535	24	3.91423	099
18	1.61045	260	68	2.41984	13624	36	5827	018
19	510	1.52831	69	6100	12725	48	4.00437	0940
20	1.62003	380	70° 0′	2.50455	11838	86 0	5276	865
21	523	1.51908	30	2729	11399	12	4.10366	792
22	1.63073	415	71 0	5073	10964	24	5736	721
23	632	1.50901	30	7490	10533	36	4.21416	653
24	1.64260	366	72 0	9982	106	48	7444	588
25	900	1.49811	30	2.62555	09683	87 0	4.33865	526
26	1.65570	237	73 0	5214	265	12	40733	466
27	6272	1.48643	30	7962	8851	24	8115	410
28	7006	029	74 0	2.70807	443	36	56190	356
29	7773	1.47397	30	3752	039	48	64765	306
30	1.68575	1.46746	75 0	6806	7641	88 0	74272	258
31	9411	077	30	9975	248	12	84785	215
32	1 70284	1.45391	76 0	2.83267	6861	24	96542	174
33	1192	44687	30	6691	480	36	5.09876	137
34	2139	43966	77 0	2.90256	106	48	25274	104
35	3125	229	30	3974	5738	89 0	43491	075
36	4150	42476	78 0	7857	378	6	54020	062
37	5217	41707	30	3.01918	024	12	65792	050
38	6326	40924	79 0	6173	4679	18	79140	049
39	7479	126	30	3.10640	4341	24	94550	030
40	1.78677	1.39314	80 0	5339	011	30	6.12778	021
41	9922	38489	12	7288	3882	36	35038	014
42	1.81216	37650	24	9280	754	42	63854	008
43	2560	36800	36	3.21317	628	48	7.04398	004
44	3957	35938	48	3400	503	54	73711	011
45	5407	35064	81 0	5530	379	90	∞	000
46	6915	34181	12	7711	257			
47	8481	33287	24	9945	126			
48	1.90108	32384	36	3.32234	017			
49	1800	31473	48	4580	2900			

Figure 2.1: Table 1

ELLIPTIC INTEGRALS OF THE FIRST KIND

ϕ	θ								
	50°	55°	60°	65°	70°	75°	80°	85°	90°
46°	0.85515	0.86431	0.87342	0.88213	0.89005	0.89678	0.90193	0.90517	0.90628
47	87614	88601	89585	90529	91390	92224	92687	93042	93163
48	89729	90791	91853	92875	93811	94610	95226	95614	95747
49	91860	93001	94146	95252	96267	97139	0.97810	0.98235	0.98381
50	94008	95232	96465	0.97660	0.98762	0.99711	1.00444	1.00909	1.01068
51	96171	97484	0.98811	1.00102	1.01297	1.02329	03129	03638	03812
52	0.98352	0.99759	1.01185	02578	03872	04995	05868	06425	06616
53	1.00550	1.02055	03587	05089	06491	07711	08665	09274	09483
54	02765	04374	06018	07637	09155	10481	11521	12188	12418
55	04998	06716	08479	10223	11865	13307	14442	15171	15423
56	07248	09082	10971	12848	14624	16190	17430	18229	18505
57	09517	11472	13494	15513	17433	19136	20488	21364	21667
58	11803	13886	16050	18220	20295	22145	23623	24582	24916
59	14108	16325	18638	20970	23212	25223	26837	27890	28257
60	16432	18788	21254	23764	26186	28371	30135	31292	31696
61	18773	21277	23916	26604	29219	31594	33524	34795	35240
62	21134	23792	26606	29490	32314	34897	37008	38407	38899
63	23513	26332	29332	32425	35473	38281	40594	42135	42679
64	25910	28898	32094	35409	38699	41753	44288	45989	46591
65	28326	31491	34893	38443	41994	45316	48098	49977	50645
66	30760	34109	37728	41529	45360	48976	52031	54112	54855
67	33212	36753	40600	44668	48800	52738	56096	58404	59232
68	35683	39423	43510	47860	52317	56606	60303	62868	63794
69	38171	42119	46457	51107	55913	60586	64661	67518	68557
70	40677	44840	49441	54410	59591	64684	69181	72372	73542
71	43200	47587	52463	57768	63352	68905	73877	77450	78771
72	45739	50359	55522	61182	67198	73256	78759	82774	84273
73	48296	53155	58618	64653	71132	77743	83844	88370	90079
74	50867	55974	61750	68180	75155	82371	89146	1.94267	1.96226
75	53455	58817	64918	71763	79269	87145	1.94682	2.00499	2.02759
76	56056	61682	68120	75401	83473	92073	2.00470	07106	09732
77	58672	64569	71356	79094	87768	1.97157	06529	14136	17212
78	61302	67476	74625	82840	92154	2.02403	12878	21644	25280
79	63943	70403	77924	86637	1.96630	07813	19538	29694	34040
80	66597	73347	81253	90484	2.01193	13390	26527	38365	43625
81	69261	76309	84609	94377	05840	19131	33866	47748	54209
82	71935	79286	87991	1.98313	10568	25035	41569	57954	66031
83	74618	82278	91395	2.02290	15371	31097	49648	69109	79422
84	77309	85281	94821	06303	20244	37309	58105	81362	2.94870
85	80006	88296	1.98264	10348	25178	43658	66935	2.94869	3.13130
86	82710	91320	2.01723	14421	30166	50129	76116	3.09782	35467
87	85418	94351	05194	18515	35198	56703	85612	26198	3.64253
88	88129	1.97388	08674	22627	40265	63357	2.95366	44116	4.04813
89	90843	2.00429	12161	26750	45354	70068	3.05304	63279	4.74135
90	1.93558	2.03472	2.15652	2.30879	2.50455	2.76806	3.15339	3.83174	∞

Figure 2.2: Table 2

$$v = \int_0^y \frac{dt}{\sqrt{f(t)}}$$

then the equation (2.2) can be represented in the form

$$u + v = A$$

where A is suitable constant.
Also, equation (2.4) can be replaced by the system

$$\frac{dx}{dt} = \sqrt{(1-x^2)(1-k^{x2})} \tag{2.5}$$

$$\frac{dy}{dt} = -\sqrt{(1-y^2)(1-k^2y^2)} \tag{2.6}$$

Squaring (2.5) and (2.6) we get

$$\left(\frac{dx}{dt}\right)^2 = (1-x^2)(1-k^2x^2) \tag{2.7}$$

$$\left(\frac{dy}{dt}\right)^2 = (1-y^2)(1-k^2y^2) \tag{2.8}$$

Let us now differentiate these equations:

$$\frac{d^2x}{dt^2} = x(2k^2x^2 - 1 - k^2)$$

$$\frac{d^2y}{dt^2} = y(2k^2y^2 - 1 - k^2)$$

this implies

$$y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2} = 2k^2xy(x^2 - y^2)$$

or,

$$\frac{d}{dt}(y\frac{dx}{dt} - x\frac{dy}{dt}) = 2k^2xy(x^2 - y^2) \tag{2.9}$$

Also, it follows from (2.7) and (2.8)

$$y^2\left(\frac{dx}{dt}\right)^2 - x^2\left(\frac{dy}{dt}\right)^2 = (y^2 - x^2)(1 - k^2x^2y^2) \tag{2.10}$$

dividing (2.9) by (2.10) we get

$$\frac{\frac{d}{dt}(y\frac{dx}{dt}-x\frac{dy}{dt})}{y\frac{dx}{dt}-x\frac{dy}{dt}}=\frac{2k^2xy(y\frac{dx}{dt}+x\frac{dy}{dt})}{k^2x^2y^2-1}$$

or,

$$\frac{d}{dt}ln\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right)=\frac{d}{dt}ln(k^2x^2y^2-1)$$

Then we have

$$y\frac{dx}{dt}-x\frac{dy}{dt}=c(k^2x^2y^2-1).$$

Taking (2.5) and (2.6) into account, we get

$$\frac{x\sqrt{(1-y^2)(1-k^2y^2)}+y\sqrt{(1-x^2)(1-k^2x^2)}}{1-k^2x^2y^2}=c.$$

We get our required algebraic form of the integral of the equation (2.4). The theorem is proved.

2.4.2. Addition theorem in jacobi elliptic function

Now, the addition theorem is better understood when we use the **Jacobi amplitude function** $\mathrm{am}(u,k)$, which is the inverse of $F(\phi,k)$. That is:

$$u=F(\phi,k) \quad \Leftrightarrow \quad \phi=\mathrm{am}(u,k)$$

Then, the **addition theorem** for elliptic integrals (or the Jacobi amplitude function) can be expressed in terms of **Jacobi elliptic functions**.

Let $u=F(\phi,k)$, $v=F(\psi,k)$. Then the addition formula for the amplitude is:

$$\mathrm{sn}(u+v)=\frac{\mathrm{sn}(u)\mathrm{cn}(v)\mathrm{dn}(v)+\mathrm{sn}(v)\mathrm{cn}(u)\mathrm{dn}(u)}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)}$$

Similarly, there are addition formulas, such as:

$$\mathrm{cn}(u+v)=\frac{\mathrm{cn}(u)\mathrm{cn}(v)-\mathrm{sn}(u)\mathrm{sn}(v)\mathrm{dn}(u)\mathrm{dn}(v)}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)}$$

$$\mathrm{dn}(u+v)=\frac{\mathrm{dn}(u)\mathrm{dn}(v)-k^2\mathrm{sn}(u)\mathrm{sn}(v)\mathrm{cn}(u)\mathrm{cn}(v)}{1-k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)}$$

Addition theorem

Set

$$F(\varphi)=\int_0^{\varphi}\frac{d\phi}{\Delta(\phi)},\quad E(\varphi)=\int_0^{\varphi}\Delta(\phi)\,d\phi,$$

where $\Delta(\varphi)=\sqrt{1-k^2\sin^2\varphi}$.
If $F(\varphi)+F(\psi)=F(\mu)$, then $\sin\mu$ can be algebraically expressed in terms of $\sin\varphi$ and $\sin\psi$. To prove this, consider the differential equation

$$\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}+\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}=0. \qquad (*)$$

It's integral is $F(\varphi)+F(\psi)-F(\mu)=0$, where μ is a constant. Taking into account that F is an odd function, the integral can be expressed in the form $F(\varphi)+F(\psi)+F(-\mu)=0$. Let us show that the integral of the differential equation (*) satisfies the relation

$$\cos\varphi\cos\psi-\sin\varphi\sin\psi\sqrt{1-k^2\sin^2\mu}=\cos\mu. \qquad (7.1)$$

This can be rewritten after squaring in a more symmetric form:

$$\cos^2\varphi\cos^2\psi+\cos^2\mu-2\cos\varphi\cos\psi\cos\mu+k^2\sin^2\varphi\sin^2\psi=1. \qquad (7.2)$$

The term "symmetric" means that not only is (7.1) satisfied but also the relations

$$\cos\varphi\cos\mu+\sin\varphi\sin\mu\sqrt{1-k^2\sin^2\psi}=\cos\psi, \qquad (7.3)$$

$$\cos\mu\cos\psi+\sin\mu\sin\psi\sqrt{1-k^2\sin^2\varphi}=\cos\varphi. \qquad (7.4)$$

Since the arguments φ,ψ,μ and $-\mu$ enter (7.2) symmetrically.

Divide both sides of (7.1) by $\sin\psi$ and differentiate the obtained expression. The result can be expressed in the form:

$$d\varphi\left(\frac{\cos\psi\cos\mu+\cos\varphi\cos\mu}{\sin\psi}\right)+d\psi\left(\frac{\cos\varphi\cos\mu+\cos\psi\cos\mu}{\sin\psi}\right)=0.$$

Making use of formulas (7.3) and (7.4) we get:

$$\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}+\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}=0.$$

Thus, (7.1) is indeed an integral of the differential equation (*). But it cannot have two independent integrals and, therefore, the equality $F(\varphi)+F(\psi)=F(\mu)$ implies that:

$$\cos\varphi\cos\psi-\sin\varphi\sin\psi\sqrt{1-k^2\sin^2\mu}=\cos\mu. \tag{7.5}$$

This provides us with an implicit expression for $\cos\mu$. It is not difficult to get an explicit expression also. Let $x=\cos\mu$, then $\sin^2\mu=1-x^2$. Relation (7.2) can be considered as a quadratic expression in x. By solving it for x, we get:

$$\cos\mu=\frac{\cos\varphi\cos\psi-\sin\varphi\sin\psi\Delta(\varphi)\Delta(\psi)}{1-k^2\sin^2\varphi\sin^2\psi}. \tag{7.5}$$

(The sign is chosen so that the formulas (7.5) and (7.1) are compatible for small φ and ψ.)

By direct algebraic transformations, we can derive from (7.5) the following expressions for $\sin\mu$ and $\Delta(\mu)$:

$$\sin\mu=\frac{\sin\varphi\cos\psi\Delta(\psi)+\sin\psi\cos\varphi\Delta(\varphi)}{1-k^2\sin^2\varphi\sin^2\psi}, \tag{7.6}$$

$$\Delta(\mu)=\frac{\Delta(\varphi)\Delta(\psi)-k^2\sin\varphi\sin\psi\cos\varphi\cos\psi}{1-k^2\sin^2\varphi\sin^2\psi}. \tag{7.7}$$

2.5. Applications

The elliptic integral of the first kind appears in:

- The period of a nonlinear pendulum,
- The analysis of elliptic curves in number theory,
- Problems involving the bending of elastic rods,
- Wave propagation in nonlinear media.

Understanding this integral provides a gateway to the rich theory of elliptic functions and their wide-ranging applications.

The elliptic integral of the first kind generalizes elementary inverse trigonometric integrals and encapsulates a rich structure through its substitution, complete form, and series expansion. These properties make it an essential tool in mathematical physics and theoretical analysis, particularly in problems involving periodic or oscillatory motion.

Chapter 3

Second and Third Kind

Introduction

Elliptic integrals are a fundamental class of special functions that naturally arise in problems involving arc lengths, the motion of particles under nonlinear constraints, and various applications in geometry and physics. Among these, the second kind plays a significant role, particularly in the evaluation of arc lengths of elliptical curves and in energy-related computations in nonlinear dynamical systems.

3.1. Second kind

The incomplete elliptic integral of the second kind is defined as:

$$E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\,d\theta,$$

where:

- ϕ denotes the amplitude,
- k is the modulus of the integral, with $0 \leq k < 1$.

This integral represents the arc length of a segment of an ellipse as a function of the angular parameter ϕ. Unlike the elliptic integral of the first kind, which is related to the inversion of elliptic functions, the second kind is closely associated with the actual geometric or physical length, making it highly relevant in practical applications.

The corresponding complete elliptic integral of the second kind is obtained by setting $\phi = \frac{\pi}{2}$, yielding:

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,d\theta.$$

This integral provides the total arc length of a quarter of an ellipse and frequently appears in

classical mechanics and electromagnetic theory, particularly in problems involving elliptical boundaries or potentials.

Also Elliptic integral of second kind is defined as

$$\begin{aligned}\int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}}dt &= \int_0^\phi \sqrt{1-k^2\sin^2\theta}d\theta \\ &= \int_o^{u_1} dn^2 u du \\ &= E(u_1) \equiv E(amu_1,k) = E(\phi,k)\end{aligned}$$

Complete elliptic integral of second kind

When $y = 1$ the above integral is said to be complete

$$\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}d\theta = \int_o^K dn^2 u du = E\left(\frac{\pi}{2},k\right).$$

As similar to elliptic integral of first kind the series expansion of second kind is as follows:

$$\frac{2}{\pi}E\left(k,\frac{\pi}{2}\right) = 1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1.3}{2.4}\right)^2 \frac{k^4}{3} - \frac{1.3.5}{2.4.6}\frac{k^6}{5}\ldots$$

Examples

- **The rectification of ellipse:** The equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b$$

The Arc length of ellipse is

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$$

by writing

$$k^2 = \frac{a^2-b^2}{a^2}, \quad x = at$$

$$s = \int_0^t \frac{(1-k^2t^2)}{\sqrt{(1-t^2)(1-k^2t^2)}}dt.$$

Finally writing $t = \sin\phi$ we have

$$s = a\int_o^\phi \Delta\phi = aE(\phi).$$

Example: Table 3 gives value of the integral

$$E = \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta \sin^2\phi}$$

when θ is known

For Example if $\theta = 78°30'$ then, E=1.05024

THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

θ	K	E	θ	K	E	θ	K	E
0°	1 57080	1.57080	50°	1.93558	1.30554	82 ° 0′	3.36987	1.02784
1	092	068	51	5386	29628	12	9457	670
2	127	032	52	7288	28695	24	3.41994	558
3	187	1.56972	53	9267	27757	36	4601	447
4	271	888	54	2.01327	26815	48	7282	338
5	379	781	55	3472	25868	83 0	3.50042	231
6	511	650	56	5706	24918	12	2884	126
7	668	495	57	8036	23966	24	5814	023
8	849	296	58	2.10466	23013	36	8837	1921
9	1 58054	114	59	3002	22059	48	3.61959	821
10	284	1.55880	60	2.15652	21106	84 0	3.65186	1.01724
11	539	640	61	8421	20154	12	8525	628
12	820	368	62	2.21319	19205	24	3.71984	534
13	1 59125	073	63	4355	18259	36	5572	443
14	457	1.54755	64	7538	17318	48	9298	354
15	814	415	65	2.30879	16383	85 0	3.83174	266
16	1.60198	052	66	4390	15455	12	7211	181
17	608	1.53667	67	8087	14535	24	3.91423	099
18	1.61045	260	68	2.41984	13624	36	5827	018
19	510	1.52831	69	6100	12725	48	4.00437	0940
20	1.62003	380	70° 0′	2.50455	11838	86 0	5276	865
21	523	1.51908	30	2729	11399	12	4.10366	792
22	1.63073	415	71 0	5073	10964	24	5736	721
23	632	1.50901	30	7490	10533	36	4.21416	653
24	1.64260	366	72 0	9982	106	48	7444	588
25	900	1.49811	30	2.62555	09683	87 0	4.33865	526
26	1.65570	237	73 0	5214	265	12	40733	466
27	6272	1.48643	30	7962	8851	24	8115	410
28	7006	029	74 0	2.70807	443	36	56190	356
29	7773	1.47397	30	3752	039	48	64765	306
30	1.68575	1.46746	75 0	6806	7641	88 0	74272	258
31	9411	077	30	9975	248	12	84785	215
32	1 70284	1.45391	76 0	2.83267	6861	24	96542	174
33	1192	44687	30	6691	480	36	5.09876	137
34	2139	43966	77 0	2.90256	106	48	25274	104
35	3125	229	30	3974	5738	89 0	43491	075
36	4150	42476	78 0	7857	378	6	54020	062
37	5217	41707	30	3.01918	024	12	65792	050
38	6326	40924	79 0	6173	4679	18	79140	049
39	7479	126	30	3.10640	4341	24	94550	030
40	1.78677	1.39314	80 0	5339	011	30	6.12778	021
41	9922	38489	12	7288	3882	36	35038	014
42	1.81216	37650	24	9280	754	42	63854	008
43	2560	36800	36	3.21317	628	48	7.04398	004
44	3957	35938	48	3400	503	54	73711	011
45	5407	35064	81 0	5530	379	90	∞	000
46	6915	34181	12	7711	257			
47	8481	33287	24	9945	126			
48	1.90108	32384	36	3.32234	017			
49	1800	31473	48	4580	2900			

Figure 3.1: Table 3

Significance and Applications

The elliptic integral of the second kind is encountered in a wide variety of mathematical and physical settings, including:

- The determination of arc lengths of ellipses and other conic sections,
- The analysis of the total energy in a nonlinear oscillatory system, such as a simple pendulum with large amplitude,
- Gravitational and electric field calculations in elliptical geometries,

In the broader theory of special functions, elliptic integrals serve as building blocks for elliptic functions and modular forms. The second kind, in particular, encapsulates important physical and geometric quantities that cannot be represented using elementary functions alone.

3.2. Third kind

Introduction

The theory of elliptic integrals encompasses three fundamental types— first, second, and third each corresponding to different aspects of the geometry and dynamics associated with elliptic curves and nonlinear systems. While the first and second kinds are primarily concerned with arc lengths and energies, the **elliptic integral of the third kind** introduces an additional parameter, known as the *characteristic*, and plays an essential role in problems involving singularities or constraints.

The *incomplete elliptic integral of the third kind* is defined by the expression:

$$\Pi(n;\phi,k) = \int_0^{\phi} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}},$$

where:

- ϕ is the amplitude,
- k is the modulus ($0 \leq k < 1$),
- n is the characteristic or parameter.

This integral generalizes the elliptic integral of the first kind by introducing a singularity at $\sin^2\theta = 1/n$, making it more complex and nuanced. Its structure arises naturally in the context of constrained motion, such as a pendulum subject to additional forces, or in electrostatics, where field lines bend around singularities or mass distributions.

The *complete elliptic integral of the third kind* is defined by taking the upper limit of integration

as $\phi = \frac{\pi}{2}$:

$$\Pi(n,k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}.$$

This form is particularly useful in analytical computations and is a core component in the theory of elliptic functions, including the construction of inverse functions and the analysis of multivalued behavior near singularities.

Applications and Significance

Elliptic integrals of the third kind appear in a wide array of applications:

- The motion of a pendulum with a driving torque or in a gravitational field with constraints,
- The calculation of electric potential and capacitance in elliptical geometries with embedded conductors,
- Problems involving geodesics on ellipsoidal surfaces,
- Analysis of integrable systems with nontrivial potentials or singular behavior.

Due to its singular structure and added parameter, the third kind elliptic integral is more intricate than the first two and serves as a critical tool in advanced analytical methods and in the study of elliptic and modular functions.

Chapter 4

Comparison

Introduction: Jacobi Elliptic Functions vs Circular Functions

Trigonometric (circular) functions like $\sin(x)$ and $\cos(x)$ are foundational in describing simple periodic motion, such as the oscillation of a mass on a spring or the projection of uniform circular motion. These functions assume a constant amplitude and period, making them well-suited for **linear systems**.

However, many physical systems such as the **nonlinear pendulum**, **wave propagation in nonlinear media**, or **charged particle motion in a magnetic field** exhibit behavior that **cannot** be accurately described by simple circular functions. This is where Jacobi elliptic functions become essential.

These functions generalize the sine and cosine functions to **nonlinear** regimes. They arise naturally when solving differential equations involving **elliptic integrals**, particularly in systems where the period of oscillation depends on the amplitude.

In essence:

- **Circular functions** are solutions to linear differential equations such as

$$y'' + y = 0.$$

- **Jacobi elliptic functions** are solutions to **nonlinear** differential equations such as

$$y'' + y(1 - 2k^2) + 2k^2 y^3 = 0$$

  where k is a parameter related to the modulus of the elliptic function.

This makes Jacobi elliptic functions powerful tools for **modeling complex periodic phenomena**, acting as nonlinear counterparts to sine and cosine.

Comparison Table

Feature	Circular Functions	Jacobi Elliptic Functions
Type	Linear	Nonlinear
Basic Functions	$\sin(x)$, $\cos(x)$	$\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$
Differential Equation	$y''+y=0$	$y''+y(1-2k^2)+2k^2y^3=0$
Period	Constant	Depends on modulus k
Amplitude	Constant	May vary with initial conditions
Range	$[-1,1]$	Varies with function and k
Origin	Circular geometry	Elliptic integral inversion
Applications	Simple harmonic motion, waves	Nonlinear oscillations, pendulum

4.1. Geometrical Interpretation

Jacobi elliptic functions extend the classical trigonometric functions from circular to elliptic geometries. These functions arise naturally when analyzing arc lengths along ellipses, much like sine and cosine are defined on the unit circle. This section explores the relationship between trigonometric functions and Jacobi elliptic functions by analyzing their geometrical interpretation, focusing on how the elliptic modulus k governs the deformation from a circle to an ellipse.

Circular Geometry and Trigonometric Functions

The unit circle is defined by the equation:

$$x^2+y^2=r^2=1$$

A point on the circle can be parameterized by the angle θ, where,

$$x=\cos\theta, \quad y=\sin\theta$$

This geometric setup gives rise to the trigonometric functions, with θ as the arc length parameter (Fig 4.1).

Elliptic Geometry and Modulus

Consider now the ellipse defined by:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let us fix the semi-minor axis $b=1$, making the semi-major axis $a>1$. The eccentricity (mod-

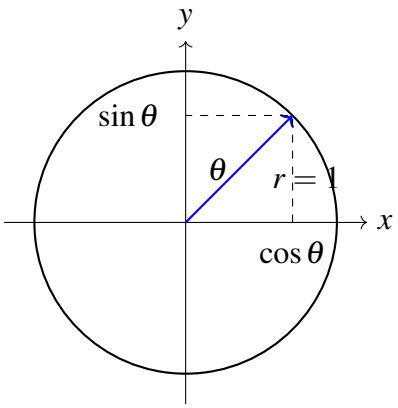

Figure 4.1: Unit circle showing projection of $x = \cos\theta$, $y = \sin\theta$

ulus) of this ellipse is:

$$k = \sqrt{1 - \frac{1}{a^2}}$$

This parameter k serves as the *modulus* in Jacobi elliptic functions, quantifying the deviation from circularity (Fig 4.2).

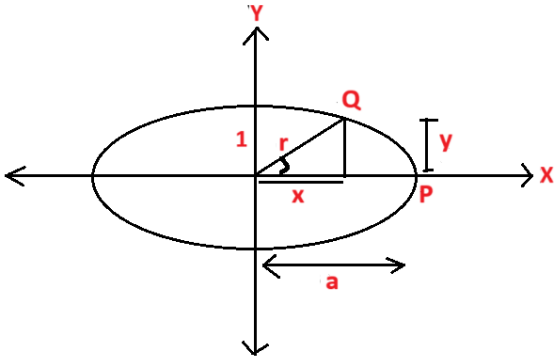

Figure 4.2: Ellipse with semi-major axis a and semi-minor axis $b = 1$

Arc Length and Elliptic Integral

The natural parameter along the ellipse is the arc length u, expressed as:

$$u = \int_P^Q r\, d\theta$$

This corresponds to the incomplete elliptic integral of the first kind and replaces the angle θ in circular geometry.

4.1.1. Jacobi Elliptic Functions from Ellipse Geometry

Using geometric definitions and normalization:

- $\mathrm{sn}(u,k) = \dfrac{y}{b} = y$, since $b = 1$

- $\mathrm{cn}(u,k) = \dfrac{x}{a}$
- $\mathrm{dn}(u,k) = \dfrac{r}{a}$

These are interpreted as:

- $\mathrm{sn}(u,k)$: vertical projection normalized by minor axis
- $\mathrm{cn}(u,k)$: horizontal projection normalized by major axis
- $\mathrm{dn}(u,k)$: radial component normalized by major axis

Trigonometric Limit as $k \to 0$

As the ellipse becomes a circle when a approaches to 1, the modulus becomes:

$$k = \sqrt{1 - \frac{1}{a^2}} \longrightarrow 0$$

In this limit:

$$r \to 1, \quad u \to \theta$$

Thus, Jacobi elliptic functions reduce to:

$$\begin{aligned} \mathrm{sn}(u, k \to 0) &= \sin\theta, \\ \mathrm{cn}(u, k \to 0) &= \cos\theta, \\ \mathrm{dn}(u, k \to 0) &= 1. \end{aligned}$$

This demonstrates the geometric connection between trigonometric and Jacobi elliptic functions. The modulus k quantifies the deviation of the ellipse from a circle. As $k \to 0$, the ellipse collapses into the unit circle, and the Jacobi elliptic functions converge to the familiar trigonometric functions, showing that the latter are a limiting case of a broader elliptic framework.

4.2. Jacobi Elliptic Functions: Derivatives and Symmetry Properties

Jacobi elliptic functions are generalizations of classical trigonometric functions and arise naturally in problems involving elliptic integrals. Defined with respect to the Jacobi amplitude function $\phi = \mathrm{am}(u)$, these functions include $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, and $\mathrm{dn}(u)$ when k is fixed, and exhibit derivative and symmetry properties that closely mirror those of sine and cosine. This section presents the derivative relationships and parity properties of these functions with trigonometric functions.

4.2.1. Definitions

Let:

$$u = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

be the incomplete elliptic integral of the first kind, where $0 < k < 1$ is the modulus. Then the Jacobi amplitude function $\phi = \mathrm{am}(u,k)$ defines the Jacobi elliptic functions as:

$$\mathrm{sn}(u,k) = \sin\phi, \quad \mathrm{cn}(u,k) = \cos\phi, \quad \mathrm{dn}(u,k) = \sqrt{1 - k^2 \sin^2 \phi}.$$

4.2.2. Derivatives

Using the chain rule and the above definitions, the derivatives of the Jacobi elliptic functions are:

$$\begin{aligned}
\frac{d}{du}\mathrm{sn}(u) &= \frac{d}{du}\sin\phi = \cos\phi \cdot \frac{d\phi}{du} = \mathrm{cn}(u) \cdot \mathrm{dn}(u) \\
\frac{d}{du}\mathrm{cn}(u) &= \frac{d}{du}\cos\phi = -\sin\phi \cdot \frac{d\phi}{du} = -\mathrm{sn}(u) \cdot \mathrm{dn}(u) \\
\frac{d}{du}\mathrm{dn}(u) &= \frac{d}{du}\sqrt{1 - k^2 \sin^2 \phi} = -k^2 \sin\phi \cos\phi \cdot \frac{d\phi}{du} = -k^2 \mathrm{sn}(u) \cdot \mathrm{cn}(u)
\end{aligned}$$

4.2.3. Initial Values at $u = 0$

If $u = 0$, then $\phi = \mathrm{am}(0) = 0$, so:

$$\mathrm{sn}(0) = \sin(0) = 0, \quad \mathrm{cn}(0) = \cos(0) = 1, \quad \mathrm{dn}(0) = \sqrt{1 - k^2 \cdot 0} = 1.$$

Hence, the initial values are:

$$\boxed{\mathrm{am}(0) = 0, \quad \mathrm{sn}(0) = 0, \quad \mathrm{cn}(0) = 1, \quad \mathrm{dn}(0) = 1}$$

4.2.4. Symmetry and Parity Properties

Jacobi elliptic functions exhibit the following symmetry (parity) properties:

Function	Symmetry Type	Identity	Trigonometric Analogy
$\mathrm{sn}(-u)$	Odd	$-\mathrm{sn}(u)$	$\sin(-u) = -\sin(u)$
$\mathrm{cn}(-u)$	Even	$\mathrm{cn}(u)$	$\cos(-u) = \cos(u)$
$\mathrm{dn}(-u)$	Even	$\mathrm{dn}(u)$	
$\mathrm{am}(-u)$	Odd	$-\mathrm{am}(u)$	

These follow from the symmetry of the integrand in the definition of the elliptic integral under sign reversal.

The Jacobi elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$ maintain structural and behavioral analogies with classical sine and cosine functions. Their derivative relationships reveal a deep interdependence, while the parity properties mirror those of trigonometric functions. These functions are central to the analysis of nonlinear oscillations, elliptic integrals, and physical systems such as pendulum motion and wave propagation in nonlinear media.

4.3. Differential Equation of Jacobi Elliptic Functions and Comparison with Circular Functions

Jacobi elliptic functions solve certain nonlinear differential equations and can be seen as extensions of standard trigonometric functions. They typically appear in situations where the restoring force is nonlinear—for example, in the motion of a pendulum swinging with large amplitudes, where the simple harmonic approximation no longer applies.

Differential Equation for Jacobi Elliptic Function $\mathrm{sn}(u,k)$

The function $\mathrm{sn}(u,k)$, one of the fundamental Jacobi elliptic functions, satisfies the nonlinear second-order differential equation:

$$\frac{\mathrm{d}^2\mathrm{sn}(u,k)}{\mathrm{d}u^2} + (1+k^2)\mathrm{sn}(u,k) - 2k^2\mathrm{sn}^3(u,k) = 0.$$

Alternatively, a more common and useful form of the first-order identity is:

$$\left(\frac{\mathrm{dsn}(u,k)}{\mathrm{d}u}\right)^2 = (1-\mathrm{sn}^2(u,k))(1-k^2\mathrm{sn}^2(u,k)).$$

This equation highlights the nonlinear nature of $\mathrm{sn}(u,k)$, and demonstrates that it interpolates between sinusoidal and hyperbolic behavior depending on the modulus k.

Comparison with Circular Functions

In the limit as $k \to 0$, Jacobi elliptic functions reduce to circular trigonometric functions. For instance,

$$\mathrm{sn}(u,0) = \sin(u), \quad \mathrm{cn}(u,0) = \cos(u), \quad \mathrm{dn}(u,0) = 1.$$

The standard sine function satisfies the linear second-order differential equation:

$$\frac{\mathrm{d}^2\sin u}{\mathrm{d}u^2} + \sin u = 0,$$

or equivalently,

$$\left(\frac{\mathrm{d}\sin u}{\mathrm{d}u}\right)^2 = 1 - \sin^2 u = \cos^2 u.$$

We can see that this is a special case of the Jacobi equation when $k = 0$, making Jacobi elliptic functions a broader class of periodic functions that encapsulate both linear and nonlinear oscillatory behavior.

Jacobi elliptic functions generalize circular functions by incorporating a parameter k that controls the degree of nonlinearity. This allows them to model a wider range of physical phenomena, such as the dynamics of nonlinear oscillators, beyond the reach of simple sine and cosine functions.

Second order non linear differential equation

- For sn(u,k)

$$\frac{d^2}{du^2}sn(u,k) + (1+k^2)sn(u,k) - 2k^2 sn^3(u,k) = 0$$

- For cn(u,k)

$$\frac{d^2}{du^2}cn(u,k) + (1+k^2)cn(u,k) - 2k^2 cn^3(u,k) = 0$$

- For dn(u,k)

$$\frac{d^2}{du^2}dn(u,k) + (2-k^2)dn(u,k) - 2dn^3(u,k) = 0$$

4.4. Periodicity

- **Perodicity**

The **Jacobi elliptic integrals** are inverse functions of the **Jacobi elliptic functions**, and they exhibit rich periodic behavior rooted in the geometry of the complex plane and the properties of elliptic curves.

The **periodicity** of these functions refers to the fact that they repeat their values over regular intervals in the complex domain, much like trigonometric functions. However, unlike sine and cosine, which are periodic over a single real period, Jacobi elliptic functions and their integrals possess **two fundamental periods**, often leading to **doubly periodic** behavior when extended to the complex plane.

These periods are closely related to the **complete elliptic integrals of the first kind**, denoted $K(k)$ and $K'(k)$, where k is the *elliptic modulus* and $K'(k)$ corresponds to the complementary modulus $k' = \sqrt{1-k^2}$.

Understanding this periodicity is essential in applications such as solving nonlinear differential equations, modeling wave propagation, and describing motion in conservative systems.

Making use of the formula
$am\,K = \dfrac{\pi}{2}$
$sn\,K = 1; cn\,K = 0; dn\,K = k'$

$$sn(u \pm 2K) = -sn\,u \quad cn(u \pm 2K) = -cnu \quad dn(u \pm 2K) = dn\,u$$

$$sn(u \pm 4K) = sn\,u \quad cn(u \pm 4K) = cn\,u \quad dn(u \pm 4K) = dn\,u$$

Note that: 4K is a period of three elliptic transcendents snu, cnu, dn u
2K is a period of dn u
Also,

$$sn\,2K = 0 \quad cn\,2K = -1 \quad dn\,2K = 1,$$

$$sn\,4K = 0 \quad cn\,4K = 1 \quad dn\,4K = 1,$$

and since K' is a function of k' as K is of k we also have

$$sn(u \pm 2K', k') = -sn(u, k')$$
$$sn(u \pm 4k', k') = sn(u, k')$$

etc.

$$cn(2mK \pm u) = cos(m\pi \pm amu) = (-1)^m cn\,u$$
$$sn(2mK \pm u) = sin(m\pi \pm amu) = (-1)^m sn\,u$$
$$dn(2mK \pm u) = dnu$$

this is analogous to

$$cos(m\pi \pm \theta) = (-1)^m cos\theta$$
$$sin(m\pi \pm \theta) = (-1)^m sin\theta$$

Summary

In this we have studied the definition of elliptic integral and their types.After that we have seen the origin and early developments in elliptic integral.Then we saw the arc length of lemniscate, elastic curve and eqation of ellipse

Then we studied the elliptic integral of first kind in which we have seen the special case when it is equal to circular function,then the series expansion of first kind. After that we have studied jacobi elliptic function,their role in first kind.After that we studied the examples of first kind,like arc length of lemniscate and time period of pendulum when θ is large. After that for computing two elliptic integral of first kind together we have seen addition theorem and other is in the form of jacobi elliptic functions.thereafter few applications.

Furthermore,we have seen about elliptic integral of second and third kind.we have studied their structures,series exapansion and examples involving the rectification of ellipse and with the use of solution table we can compute the value for second kind.

After that we have seen the comparission between jacobi elliptic function and circular function,in which we have started from their differential equation.we were compare them with the help of table. After that we have seen the geometric interpretation of jacobi elliptic function in which we have started from circular geometry and elliptic geometry .After that we have studied the comparission between differential equation of jacobi elliptic function and circular function.After that we have studied the periodicity of jacobi elliptic function and their analogous to circular function.

Future Works

- Explore the different types of elliptic functions such as Weirstrass Elliptic Function.
- Study more Examples on the Elliptic Integral of Third Kind.

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